Nicholas Farkash - Assignment 15

1 Introduction

In Assignment 10, we used Poisson's Equation, a partial different equation, for a 2-D steady state electromagnetics problem to determine the electric potential of a system at a particular point. In this document, the process of determining this value will be briefly explored.

2 Equation

Poisson's Equation is a partial differential equation. When applied to our two-dimensional electomagnetics problem, it takes the form:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 4\pi q \tag{1}$$

where U is the electrical potential and q is the electrical charge density.

In word form, this equation says that the second derivative of the electric potential with respect to the x direction plus the second derivative of the electric potential with respect to the y direction is equal to 4π times the electrical charge density.

2.1 Finite-Difference Form

If we divide the box we are observing into square-shaped cells of sides h, we can refer to each individual cell by its location $U_{i,j}$, which we can then use to loop over in our calculations. By approximating the first derivatives of the electric potential at the edge of each cell, and then the second derivatives of the electric potential at the center of each cell, we can calculate the charge density of any particular cell.

The first derivatives are approximated as follows:

$$\left(\frac{\partial U}{\partial x}\right)_{i,j+\frac{1}{2}} \approx \frac{U_{i,j+1} - U_{i,j}}{h} \qquad \left(\frac{\partial U}{\partial y}\right)_{i+\frac{1}{2},j} \approx \frac{U_{i+1,j} - U_{i,j}}{h} \tag{2}$$

The second derivatives are approximated as follows:

$$\left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} \approx \frac{\left(\frac{\partial U}{\partial x}\right)_{i,j+\frac{1}{2}} - \left(\frac{\partial U}{\partial x}\right)_{i,j-\frac{1}{2}}}{h} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2} \tag{3}$$

$$\left(\frac{\partial^2 U}{\partial y^2}\right)_{i,j} \approx \frac{\left(\frac{\partial U}{\partial y}\right)_{i+\frac{1}{2},j} - \left(\frac{\partial U}{\partial y}\right)_{i-\frac{1}{2},j}}{h} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} \tag{4}$$

We can insert equations (3) and (4) into equation (1) and produce the following equation:

$$\frac{U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}}{h^2} = 4\pi q_{i,j} \tag{5}$$

Solving equation (5) for $U_{i,j}$ yields:

$$\frac{U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4\pi h^2 q_{i,j}}{4} = U_{i,j} \tag{6}$$

This formula takes the sum of the values of the surrounding cells, subtracts the value $4\pi h^2 q_{i,j}$, and divides this by 4 to approximate the new value.

3 Procedure

Using equation (6), we can write a program to calculate the electric potential. The iterative process used is called the relaxation method, and the procedure for this method is shown below:

Begin with equation (6):

1. Make an initial guess for the solution at each cell. For simplicity, make this guess 0 for all cells.

2. Use these initial guesses on the right side of equation (6).

3. Solve for $U_{i,j}$. This is the new estimate of the solution.

4. Use the new estimate on the right side of equation (6).

5. Repeat this process until the values stop changing (or are changing so slightly that they are negligible). This means the solution has converged to the answer.

This method works well for the interior cells of the box, but what about the cells on the boundary? When using the relaxation method on these cells, there may not be a cell adjacent to the box whose value can be used during the calculation. To avoid this issue, we create a bigger box around the first box that is one cell larger in each direction. Since these new cells are outside of the box containing the charge, they will all always have the value 0. This gives each cell inside the box all four adjacent cells necessary to carry out the relaxation method's calculations.

4 Results

For Assignment 10, we were asked to find the electric potential in the cell (25,50). The value at this particular cell was 6.18657880E-02 CGS units.

Below is an image of a heat map, which depicts the converged solutions of Assignment 10.

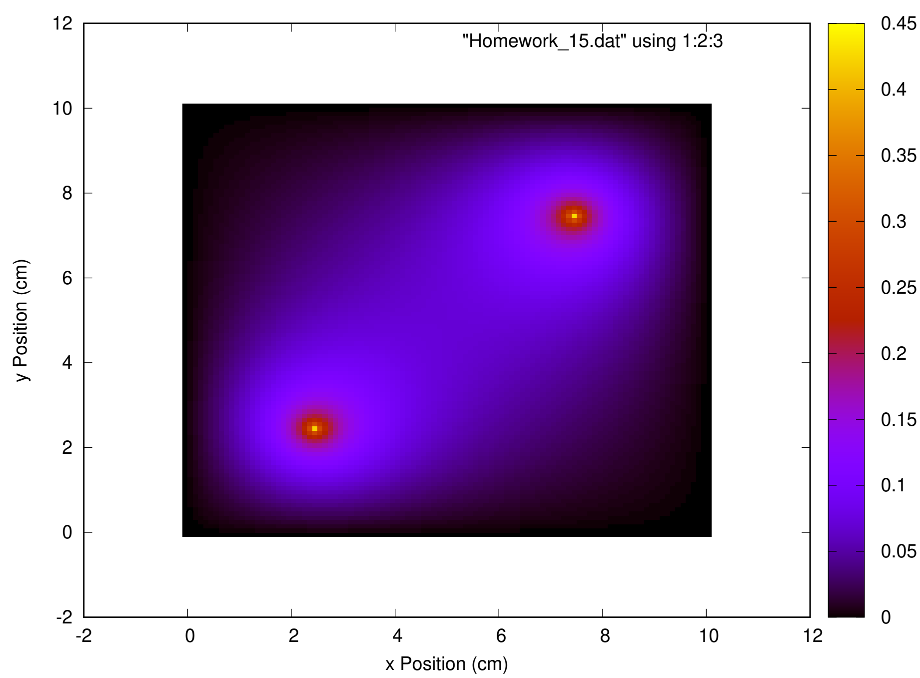

Figure 1: A heat map of the results